

# EXTREME VALUE DISTRIBUTION FOR NORMALIZED SUMS FROM STATIONARY GAUSSIAN SEQUENCES

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Let  $\{X_i, i \geq 0\}$  be a sequence of independent identically distributed random variables with finite absolute third moment. Then Darling and Erdős have shown that

$$\lim_{n \rightarrow \infty} P\left(\mu_n \leq \chi_n + \frac{\ln_3 n - \ln 4\pi}{2\chi_n} + \frac{t}{\chi_n}\right) = \exp(-e^{-t})$$

for  $-\infty < t < \infty$  where  $\mu_n = \max_{0 \leq k \leq n} k^{-1/2} \sum_{i=0}^k X_i$  and  $\chi_n = (2 \ln \ln n)^{1/2}$ . The result is extended to dependent sequences but assuming that  $\{X_i\}$  is a standard stationary Gaussian sequence with covariance function  $\{r_i\}$ . When  $\{X_i\}$  is moderately dependent (e.g. when  $v(\sum_{i=1}^n X_i) \sim n^\alpha$ ,  $0 < \alpha < 2$ ) we get

$$\lim_{n \rightarrow \infty} P\left(\mu_n \leq \chi_n + \left(\frac{1}{\alpha} - \frac{1}{2}\right) \frac{\ln\left(\frac{H_\alpha}{2\pi^{1/2}} \ln \ln n\right)}{\chi_n} + \frac{t}{\chi_n}\right) = \exp(-e^{-t})$$

where  $H_\alpha$  is a constant. In the strongly dependent case (e.g. when  $v(\sum_{i=1}^n X_i) \sim n^2 r(n)$ ) we get

$$\lim_{n \rightarrow \infty} P\left(\mu_n \leq \chi_{1/r_n} + \frac{1}{2} \frac{\ln_3 1/r_n - \ln 4\pi}{\chi_{1/r_n}} + \frac{t}{\chi_{1/r_n}}\right) = \exp(-e^{-t})$$

for  $-\infty < t < \infty$ .

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## 1. Introduction

Let  $\{X_i, i \geq 0\}$  be a sequence of independent identically distributed random variables with mean zero and variance one. Many classical problems have come out of the study of the partial sums  $S_k = \sum_{i=1}^k X_i$ . The celebrated law of the iterated

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logarithms (LIL) gives the first order terms in the growth of  $S_k$ . It states

$$\lim_{k \rightarrow \infty} \frac{S_k}{(2k \ln \ln k)^{1/2}} = 1 \quad \text{a.s.} \quad (1)$$

Sometimes the LIL is stated in the following Feller form, giving information about the second order terms. For  $\phi(n)$  positive and nondecreasing

$$P(S_k > k^{1/2} \phi(k) \text{ i.o.}) = \begin{cases} 0, & \text{if } \sum \frac{\phi(n)}{n} \exp(-\frac{1}{2}\phi^2(n)) < \infty, \\ 1, & \text{if } \sum \frac{\phi(n)}{n} \exp(-\frac{1}{2}\phi^2(n)) = \infty. \end{cases} \quad (2)$$

The quantity  $S_k/k^{1/2}$  in both (1) and (2) can very easily be replaced by  $\max_{1 \leq i \leq k} S_i/i^{1/2}$ . In this paper we will be interested in the behavior of this maximum. Besides the fixed growth rate given so precisely by (2), the randomness of  $\max_{1 \leq i \leq k} S_i/i^{1/2}$  is also of interest. Darling and Erdős [3] prove that if  $\{X_i\}$  has finite absolute third moment, then

$$\lim_{n \rightarrow \infty} P \left\{ \max_{1 \leq k \leq n} \frac{S_k}{k^{1/2}} \leq (2 \ln_2 n)^{1/2} + \frac{\ln(\ln_2 n/4\pi)}{2(2 \ln_2 n)^{1/2}} + \frac{t}{(2 \ln_2 n)^{1/2}} \right\} = \exp(-e^{-t}) \quad (3)$$

for all  $-\infty < t < \infty$ . As usual  $\ln_q n = q$ th iterated logarithm of  $n$ .

The invariance principle of Erdős and Kac offers an elegant tool to prove these results. The required results are first proved in the special case when  $X_i$  are normally distributed. The Central Limit Theorem lets us approximate partial sums  $S_k/(k^{1/2})$  by normal variabiles. The results then follow by essentially concluding that this approximation is, in fact, very good. Thus the study of the partial sums from the stationary Gaussian sequences is more significant than any other particular case in this area.

From now on, let  $\{X_i\}$  be assumed to be Gaussian. It is interesting to see how (2) and (3) are affected when the assumption of independence among  $X_i$  is relaxed. Theorem 4 of Lai and Stout [5] states the following.

**Theorem (Lai and Stout).** *Suppose there exists  $0 < \alpha \leq 2$ ,  $\gamma > \alpha/2$ ,  $M \geq 1$ ,  $\delta > 0$ ,  $\beta > 0$  and positive sequence  $\{L(n)\}$  satisfying the following conditions:*

(i)

$$n^\alpha L(n) \ll v(n) \ll n^\alpha L(n) \quad \text{as } n \rightarrow \infty, \quad v(n) = \text{Var} \left( \sum_{i=1}^n X_i \right),$$

(ii)

$$\left| \frac{v(n+m)}{v(n)} - 1 \right| \leq M \left( \frac{m}{n} \right)^\gamma \quad \text{if } \delta n \geq m \geq M,$$

(iii)

$$\limsup_{n \rightarrow \infty} \left\{ \max_{n(\ln_2 n)^{-\beta} \leq j \leq n} \frac{L(j)}{L(n)} \right\} < \infty \quad (4)$$

and

$$\liminf_{n \rightarrow \infty} \left\{ \min_{n(\ln_2 n)^{-\beta} \leq j \leq n} \frac{L(j)}{L(n)} \right\} > 0,$$

iv)  $\forall \rho > 0$  there exists  $m_\rho$  and  $1 \geq \lambda_\rho > 0$  such that if  $\lambda_\rho n \geq m \geq m_\rho$ , then

$$\left( \frac{m}{n} \right)^\rho \leq \frac{L(n)}{L(m)} \leq \left( \frac{n}{m} \right)^\rho.$$

Let  $\phi(t)$  be a positive nondecreasing function on  $[1, \infty)$ . Then

$$P(S_n > (v(n))^{1/2} \phi(n) \text{ i.o.}) = \begin{cases} 0, & \text{if } \int_1^\infty \frac{(\phi(t))^{2/\alpha-1}}{t} \exp\left(-\frac{\phi^2(t)}{2}\right) dt < \infty \\ 1, & \text{if } \int_1^\infty \frac{(\phi(t))^{2/\alpha-1}}{t} \exp\left(-\frac{\phi^2(t)}{2}\right) dt = \infty. \end{cases} \quad (5)$$

In Sections 2 and 3 we extend (3) in the cases of moderately and strongly dependent stationary Gaussian sequences. We prove that if  $\mu_n = \max_{0 \leq i \leq n} S_i / (v(i))^{1/2}$  and

$$v(n) \sim n^\alpha L(n) \quad \text{as } n \rightarrow \infty \quad (6)$$

with some additional conditions on the slowly varying function  $L(n)$  to make it well behaved, then

$$\lim_{n \rightarrow \infty} P(\mu_n \leq (2 \ln_2 n)^{1/2} + \left( \frac{1}{\alpha} - \frac{1}{2} \right) \frac{\ln(c_\alpha \ln_2 n)}{(2 \ln_2 n)^{1/2}} + \frac{x}{(2 \ln_2 n)^{1/2}}) = \exp(-e^{-x}) \quad (7)$$

for all  $-\infty < x < \infty$ . Also if

$$v(n) \sim \left( 1 + o\left( \frac{1}{(\ln n)^\delta} \right) \right) n^2 H(n) \quad (8)$$

for some  $\theta > 0$  again with some additional conditions on  $r(n)$ , then

$$\lim_{n \rightarrow \infty} P\left(\mu_n \leq (2 \ln_2 1/r_n)^{1/2} + \frac{1}{2} \frac{\ln_2 1/r_n - \ln 4\pi}{(2 \ln_2 1/r_n)^{1/2}} + \frac{x}{(2 \ln_2 1/r_n)^{1/2}}\right) + \exp(-e^{-x}) \quad (9)$$

for all  $-\infty < x < \infty$ .

The proofs as in [3], compare the maximum  $\mu_n$  to that of a suitably chosen stationary Gaussian process on an appropriate set. The sketch of the main idea is as follows. We know that even though  $\{S_k/(v(k))^{1/2}\}$  forms a standard Gaussian sequence, it is no longer stationary. However, we could view this as coming from a stationary process being sampled with increasing frequency. If  $v(k) \sim k^\alpha L(k)$ ,  $0 < \alpha < 2$ , then sample from an  $\alpha$ -process (i.e., the correlation function near the origin is  $\sim 1 - \frac{1}{2}|t|^\alpha$ ) at points  $1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, \dots, \sum_{i=1}^n 1/i, \dots$ . Thus we have  $n$  observations for this  $\alpha$ -process in the interval about  $[0, \ln n]$ . The maximum of these  $n$  observations closely approximates  $\mu_n$ . In this, what we call a 'moderately dependent case', the frequency of sampling does not depend on the covariance function of  $\{X_i\}$ .

It is interesting to note that the above procedure fails when  $\alpha = 2$ . Oddly enough, we observe that in this case, the normalized sums process can be approximated by a Brownian motion with transformed time axis. If we have to carry the above analogy in this case as well, then we have to sample an  $\alpha = 1$  process at a frequency that depends on the covariance function of the  $X$ -process. At any point  $k$ , instead of placing the next observation at a distance  $1/k$  from the  $k$ th observation, we place it at a distance approximately  $|r'(k)/r(k)|$ . We will call this the 'strongly dependent' case. Thus, in this case,  $\mu_n$  approximates the maximum of a  $\alpha = 1$  process on the set  $[0, \ln 1/r_n]$ . This allows arbitrarily slow growth rates of  $\mu_n$  by choice of  $r_n$ . Though not surprising, it is interesting to note that in all cases, only the double exponential is obtained as the extreme value distribution.

There may seem a discrepancy between (5) and (9). According to (5), if  $v(k) = k^2 L(k)$  satisfies (4), then the first order term for  $\mu_n$  is  $(2 \ln_2 n)^{1/2}$  while as (9) places it at a much smaller number viz  $(2 \ln_2 1/r_n)^{1/2}$ . None of the covariances looked at in (8) satisfy (4). In fact, we could not find any examples of covariance functions satisfying (4) for  $\alpha = 2$ .

The next two sections give exact statements and the proofs of the moderately and the strongly dependent cases respectively. Examples of covariance functions satisfying the given conditions and some comments are in Section 4.

## 2. Moderately dependent sequences

Throughout the remaining,  $\{X_i, i \geq 0\}$  will be a stationary Gaussian sequence with mean zero, variance one and  $r_i = \mathbb{E}X_0 X_i$ . Let  $v(k) = \text{Var} \sum_{i=0}^k X_i$ ;  $Y_k = (v(k))^{-1/2} \sum_{i=0}^k X_i$  for  $k = 0, 1, 2, \dots$  and  $\mu_n = \max_{0 \leq k \leq n} Y_k$ .

**Theorem 1.** Suppose that for sufficiently large  $k$ ,

$$\sum_{i=-k}^k r_i = k^\delta L(k) > 0, \quad (10)$$

$-1 < \delta < 1$  and  $L(k)$  is a slowly varying function satisfying

$$\begin{aligned} & \text{(i) for sufficiently large } l, L(k)/L(l) \text{ is bounded if } l \geq k \geq l^\theta, \theta > 0 \text{ and} \\ & \text{(ii) if } l \geq k, k \rightarrow \infty \text{ such that } (l-k)/k^{1-\gamma} \rightarrow \infty \forall \gamma > 0 \text{ but } (l-k)/k \rightarrow c \\ & \text{for } 0 \leq c \leq \infty, \text{ then} \end{aligned} \quad (11)$$

$$\lim_{k \rightarrow \infty} \frac{l}{k} \frac{L(l) - L(l-k)}{L(l-k)} = 0.$$

Then for  $\alpha = \delta + 1$ ,

$$\lim_{n \rightarrow \infty} P\left(\mu_n \leq \beta_n + \frac{x}{\lambda_n}\right) = \exp(-e^{-x}) \quad (12)$$

for all  $-\infty < x < \infty$ . We write

$$\chi_n = (2 \ln_2 n)^{1/2}; \quad \beta_n = \chi_n + \frac{(1/\alpha - \frac{1}{2}) \ln_3 n + \ln(H_\alpha/\sqrt{4\pi})}{\chi_n}$$

and

$$H_\alpha = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^\infty e^u P\left(\max_{0 \leq t \leq T} \zeta(t) > u\right) du$$

where  $\zeta(t)$  is a separable nonstationary Gaussian process with  $\mathbf{E}\zeta(t) = -|t|^\alpha$  and  $\text{Cov}(\zeta(s), \zeta(t)) = |s|^\alpha + |t|^\alpha - |s-t|^\alpha$ .

Before we proceed to prove (12), we will state and prove two lemmas about  $\mathbf{E}Y_k Y_l$  which will simplify the proof of Theorem 1.

**Lemma 1.** For  $[\exp(\varepsilon m)] \leq k < l < [\exp(\varepsilon(m+1))]$ ,  $\varepsilon > 0$  and  $m$  sufficiently large.

$$1 - (\tfrac{1}{2} + \varepsilon') \frac{v(l-k)}{v(k)} \leq \mathbf{E}Y_k Y_l \leq 1 - (\tfrac{1}{2} - \varepsilon') \frac{v(l-k)}{v(k)}. \quad (13)$$

By  $\varepsilon'$  we denote a small positive number depending on  $\varepsilon$ , not necessarily the same at all places. Also  $[\cdot]$  denotes integral part.

**Proof.** Since  $v(k) = \sum_{j=0}^{k-1} \sum_{i=-j}^j r_i$ , (10) and Theorem 1 in ([4, p. 273]) gives that  $v(k)$  is regularly varying with exponent  $\alpha = \delta + 1$  and

$$\frac{1/\alpha k^\alpha L(k)}{v(k-1)} \rightarrow 1 \quad \text{as } k \rightarrow \infty. \quad (14)$$

Define  $\omega_{kl} = \sum_{j=1}^{l+k} \sum_{i=j}^{j+k} r_i$ . Then

$$\begin{aligned} \mathbf{E} Y_k Y_l &= \frac{v(k) + \omega_{kl}}{(v(k)v(l))^{1/2}} \\ &= 1 - \frac{(v(k)v(l))^{1/2} - (v(k) + \omega_{kl})}{(v(k)v(l))^{1/2}} \\ &= 1 - \frac{v(k)v(l) - (v(k) + \omega_{kl})^2}{v(k)v(l)\{1 + (v(k) + \omega_{kl})(v(k)v(l))^{-1/2}\}}. \end{aligned} \quad (15)$$

Notice that  $v(l) = v(k) + v(l-k) + 2\omega_{kl}$  and define  $E_{kl} = (v(l-k) + 2\omega_{kl})/v(k)$ . Then the R.H.S. above is equal to

$$\begin{aligned} &1 - \frac{v(l-k) - \omega_{kl}^2/v(k)}{v(l)\{1 + (v(k) + \omega_{kl})(v(k)v(l))^{-1/2}\}} = \\ &= 1 - \frac{v(l-k)}{v(k)} \left\{ 1 - \frac{\omega_{kl}^2}{v(k)v(l-k)} \right\} (1 + E_{kl})^{-1} \left( 1 + (1 + E_{kl})^{-1/2} + \frac{\omega_{kl}}{(v(k)v(l))^{1/2}} \right). \end{aligned} \quad (16)$$

We need to show that the product of the last three terms in (16) could be bounded above and below by  $\frac{1}{2} \pm \varepsilon'$  as  $k \rightarrow \infty$ . At (a), (b) and (c) below we find bounds for various quantities involved. First we look at  $v(l-k)/v(k)$ . The argument used here is repeated several times during the rest of the text. We have  $0 \leq l-k \leq \varepsilon'k$ . If  $(l-k) \leq k^\theta$  for some  $0 < \theta < 1$ , then we can write  $v(l-k) \leq L(l-k)k^{\alpha\theta}$  for large  $k$  and

$$\frac{v(l-k)}{v(k)} \leq \frac{L(l-k)}{L(k)k^{\alpha(1-\theta)}}.$$

Since  $L(t)$  is a slowly varying function, we know that  $L(t)t^{-\gamma} \rightarrow 0$  and  $L(t)t^\gamma \rightarrow \infty$  as  $t \rightarrow \infty$  for all  $\gamma > 0$ . If  $k^\theta < l-k \leq \varepsilon'k$ , then by (11(i)),  $L(l-k)/L(k)$  is bounded. Thus

(a)

$$\frac{v(l-k)}{v(k)} \leq \varepsilon'$$

for all  $k$  sufficiently large. Next, we will look at  $\omega_{kl} = \sum_{j=1}^{l-k} \sum_{i=j}^{j+k} r_i$ . For large  $k$ , by (10) we have

$$|\omega_{kl}| \leq (\text{const.})(l-k)l^\delta L(l).$$

Hence

$$\frac{|\omega_{kl}|^2}{v(k)v(l-k)} \leq (\text{const.}) \left( \frac{l-k}{k} \right)^{1-\delta} \frac{L^2(l)}{L(k)L(l-k)}.$$

The fact that we need  $(l-k)$  sufficiently large to make the approximation  $v(l-k) \sim (l-k)^{1+\delta} L(l-k)$  is inconsequential here, since, if  $(l-k)$  is too small, then the L.H.S. above is obviously small for  $k$  large. If  $(l-k) \leq k^\theta$ , then the R.H.S. above tends to zero as  $k \rightarrow \infty$  by same argument as at (a) and if  $(l-k) > k^\theta$  then  $L(l)/L(l-k)$  is bounded by (11(i)) and  $L(l)/L(k)$  is always bounded for the range of  $l$  and  $k$  under consideration. Thus

(b)

$$\frac{|\omega_{kl}|^2}{v(k)v(l-k)} \leq \varepsilon'$$

for all  $k$  sufficiently large. Obviously  $|\omega_{kl}|/(v(k)v(l))^{1/2} \leq \varepsilon'$  as well. Finally,

(c)

$$\frac{|\omega_{kl}|}{v(k)} \leq \frac{l-k}{k} \left(\frac{l}{k}\right)^\delta \frac{L(l)}{L(k)} \leq \varepsilon$$

for all  $k$  sufficiently large. Substituting in (16) we get the result.

**Lemma 2.** If  $l \geq k \rightarrow \infty$  such that  $l-k \geq l^\theta$  for some  $\theta > 0$ , then

$$\lim_{k \rightarrow \infty} \frac{|\omega_{kl}|}{(v(k)v(l))^{1/2}} \left(\frac{l}{k}\right)^\gamma = 0 \quad (17)$$

for some  $\gamma > 0$ .

**Proof.** Suppose  $k$  is such that  $(l-k)/k^\gamma \rightarrow 0$  for some  $0 < \gamma < 1$ . Then write

$$\begin{aligned} |w_{kl}| &= \left| \sum_{j=0}^{l-k-1} \sum_{i=-(k+j)}^{k+j} r_i - v(l-k) \right| \\ &\leq \sum_{j=0}^{l-k-1} (k+j)^\delta L(k+j) \leq (\text{const.}) l^{\delta'} (l-k) \end{aligned}$$

for all  $\delta' > \delta$  since  $L$  is a slowly varying function and  $k$  is large so that  $l/k \rightarrow 1$ . Thus

$$\begin{aligned} \frac{|w_{kl}|}{(v(k)v(l))^{1/2}} &\leq (\text{const.}) \frac{l^{\delta'} (l-k)}{k^{\alpha/2} l^{\alpha/2} (L(k)L(l))^{1/2}} \\ &< (\text{const.}) \frac{(l-k)^{(1-\delta)/2}}{l^{(1-\delta)/2 - a_0}} (L(k)L(l))^{-1/2} \end{aligned}$$

for all  $a_0 > 0$ . Choosing  $0 < a_0 < \gamma$ , we notice that the R.H.S. above is  $o(1)$ . Hence (17) is valid. Now assume that  $(l-k)/k^\gamma \rightarrow \infty$  for all  $0 < \gamma < 1$ . We can write

$$\begin{aligned} |w_{kl}| &= \left| \sum_{j=1}^k \sum_{i=-(l-j)}^{(l-j)} r_i - v(l-k) \right| \\ &\leq \sum_{j=1}^k (l-j)^\delta L(l-j) = \sum_{j=(l-k)}^{l-1} j^\delta (L(j) - L(l-k)) + L(l-k) \sum_{j=(l-k)}^{l-1} j^\delta. \end{aligned}$$

By (11(ii)), the R.H.S. above is

$$\leq \left(\frac{k}{l} + 1\right) L(l-k) \sum_{j=(l-k)}^{l-1} j^{\delta}.$$

Thus

$$\begin{aligned} \frac{|\omega_{kl}|}{(v(k)v(l))^{1/2}} &\leq (\text{const.}) \frac{kl^{\delta}L(l-k)}{(kl)^{\alpha/2}(L(k)L(l))^{1/2}} \\ &= (\text{const.}) \left(\frac{k}{l}\right)^{(1-\delta)/2} \frac{L(l-k)}{(L(k)L(l))^{1/2}}. \end{aligned}$$

Choose  $0 < \gamma < (1-\delta)/2$ . Then the R.H.S. above is  $o((k/l)^{\gamma})$  as  $k \rightarrow \infty$  because here both  $L(l-k)/L(k)$  and  $L(l-k)/L(l)$  are bounded due to (11(i)) (Notice that the case  $\delta = 1$  is excluded).

We now turn to the proof of Theorem 1.

**Proof of Theorem 1.** The proof is split into two parts. First we establish that

$$\liminf_{n \rightarrow \infty} P\left(\mu_n \leq \beta_n + \frac{x}{\chi_n}\right) \geq \exp(-e^{-x}) \quad (18)$$

for  $-\infty < x < \infty$ . We recall that  $\mu_n = \max_{0 \leq k \leq n} Y_k$  where  $Y_k = (v(k))^{-1/2} \sum_{i=1}^k \chi_i$ . The constants  $\beta_n$  and  $\chi_n$  are defined after (12).

We will split  $\{Y_i, 1 \leq i \leq n\}$  into blocks  $t_m \leq i < t_{m+1}$ ,  $m = 0, 1, \dots, m_0$  where  $t_m = [\exp(\varepsilon m)]$  for some  $\varepsilon > 0$  and  $m_0$  is such that  $t_{m_0} \approx n$ . We will find a lower bound for the probability in L.H.S. of (18) by treating  $Y_i$  in different blocks as independent since  $\mathbf{E} Y_k Y_l$  will be shown to be nonnegative. Now, suppose  $\mathbf{E} Y_k Y_l$  was in fact equal to  $1 - \frac{1}{2}((l-k)/k)^{\alpha}$ . Let  $\xi(t)$  be a standard stationary Gaussian process with covariance function  $1 - (\frac{1}{2} + \varepsilon)|t|^{\alpha}$  for  $0 \leq t \leq \varepsilon$ . We can sample  $\{\xi(t), 0 \leq t \leq \varepsilon\}$  at points  $s_i$  where  $s_1 = 1/t_m$  and  $s_k = s_{k-1} + 1/s_{k-1}$ . It would be easy to see that  $\max_k \xi(s_k)$  provides a stochastic upper bound for  $\max_{t_m \leq k < t_{m+1}} Y_k$ . However, by using Lemma 1, we can get explicit bounds for  $\mathbf{E} Y_k Y_l$  only when  $(l-k)$  is large. Thus it is necessary to delete all but a subsequence  $\{Y_{t_m, h}\}$  of variables from the  $m$ th block. The proof will follow this general outline. To start off, we also need to exclude  $Y_1, \dots, Y_{(l-n)^{1/2}}$  from consideration, since all the bounds gotten in Lemmas 1 and 2 work only for large  $k$ .

For the first part of the proof, define  $u_n = \beta_n + x/\chi_n$ . Then for  $\mu'_n = \max_{(\ln n)^{1/2} \leq k \leq n} Y_k$

$$|P(\mu_n \leq u_n) - P(\mu'_n \leq u_n)| \leq P\left(\max_{0 \leq k \leq (\ln n)^{1/2}} Y_k > u_n\right) \leq (\ln n)^{1/2} \phi(u_n)/u_n$$

where  $\phi(u) = (2\pi)^{-1/2} \exp(-u^2/2)$ . The R.H.S. above tends to zero. Thus we can replace  $\mu_n$  by  $\mu'_n$  in (12).



We will prove (18) with  $\mu_n$  replaced by  $\mu'_n$ . For  $k$  and  $l$  large  $\mathbf{E} Y_k Y_l \geq 0$  since by (10),

$$v(k) + 2\omega_{kl} > 0 \Rightarrow \omega_{kl} > \frac{-v(k)}{2} \quad \text{and} \quad v(k) + \omega_{kl} > \frac{v(k)}{2} > 0.$$

This, in view of (15) shows that  $\mathbf{E} Y_k Y_l \geq 0$  for  $k$  and  $l$  large. By Slepian's Lemma [11],

$$P(\mu'_n \leq u_n) \geq \prod_{m=m_1}^{m_0} P\left(\max_{t_m \leq l < t_{m+1}} Y_l \leq u_n\right) \quad (19)$$

where  $m_1$  is such that  $t_{m_1} \sim (\ln n)^{1/2}$ .

The next step is to select a proper subsequence  $t_{m,k}$  from the  $m$ th block. Define  $t_{m,h} = [\exp(\varepsilon(m^2 + h)^{1/2})]$  for  $h = 0, 1, \dots, (2m+1)$  i.e.,  $t_{m,0} = t_m$  and  $t_{m,2m+1} = t_{m+1}$ . Define  $Z_l = Y_l - Y_{t_{m,h}}$  for  $t_{m,h} < l \leq t_{m,h+1}$ ,  $h = 0, 1, \dots, 2m$ . Then

$$\begin{aligned} P\left(\max_{t_m \leq l < t_{m+1}} Y_l \leq u_n\right) &\geq P\left(\max_{0 \leq h \leq 2m} Y_{t_{m,h}} \leq u_n - \frac{\ln m}{m^{\alpha'/2}}\right) \\ &\quad - P\left(\max_{t_m < l < t_{m+1}} Z_l > \frac{\ln m}{m^{\alpha'/2}}\right) \end{aligned} \quad (20)$$

where  $\alpha' < \alpha$ . We will get an upper bound for the second term in the R.H.S. above by using the following result.

**Theorem 2** (Marcus and Shepp [6]). *Let  $\{W_i, i \geq 1\}$  be a sequence of jointly Gaussian random variables with  $P\{\sup_{i \geq 1} |W_i| < \infty\} = 1$ . Then, letting  $\sigma^2 = \sup_{i \geq 1} \text{Var}(W_i)$ , for  $\rho > 0$  and all sufficiently large  $t$ , we have*

$$P\left(\sup_{i \geq 1} |W_i| > t\right) < \exp\{-(1-\rho)t^2/(2\sigma^2)\}. \quad (21)$$

It is clear that  $P(\max_{t_m \leq l < t_{m+1}} Z_l < \infty) = 1$ , since  $[t_m, t_{m+1})$  contains only finite number of points. Before applying (21), we note that

$$\mathbf{E}(Y_k Y_l) \geq 1 - \left(\frac{1}{2} + \varepsilon'\right) \left(\frac{l-k}{k}\right)^\alpha \frac{L(l-k)}{L(k)} \geq 1 - \varepsilon' \left(\frac{l-k}{k}\right)^{\alpha'}. \quad (22)$$

We write the first inequality above for the sake of convenience. For  $(l-k)$  too small, we may not be able to write  $v(l-k) \sim (l-k)^\alpha L(l-k)$ . However, the second inequality is correct for large  $k$  since we have chosen  $\alpha' < \alpha$ . The argument for other values of  $(l-k)$  is similar to the one used before. Now from (21) we have

$$P\left(\max_{t_m \leq l < t_{m+1}} Z_l > \frac{\ln m}{m^{\alpha'/2}}\right) \leq \exp\left\{-\left(\frac{1-\rho}{2}\right) \left(\frac{\ln m}{m^{\alpha'/2}}\right)^2 \frac{1}{\sigma_m^2}\right\} \quad (23)$$

where  $\sigma_m^2 = \max_{t_m \leq l < t_{m+1}} \text{Var}(Z_l)$ . But for  $t_{m,h} \leq l < t_{m,h+1}$ ,

$$\begin{aligned} \text{Var}(Z_l) &= v(Y_l - Y_{t_{m,h}}) = 2(1 - \mathbf{E} Y_l Y_{t_{m,h}}) \leq 2\varepsilon' \left( \frac{l - t_{m,h}}{t_{m,h}} \right)^{\alpha'} \quad \text{using (22)} \\ &\leq \varepsilon' \left( \frac{t_{m,h+1} - t_{m,h}}{t_{m,h}} \right)^{\alpha'} \leq \varepsilon' m^{-\alpha'} \quad \forall h = 0, 1, \dots, 2m. \end{aligned}$$

Hence the R.H.S. of (23) is at most

$$\exp\{-(\text{const.})(\ln m)^2\}$$

and the R.H.S. of (19) is at least

$$\begin{aligned} \prod_{m=m_1}^{m_0} \left\{ P\left( \max_{0 \leq h \leq 2m} Y_{t_{m,h}} \leq u_n - \frac{\ln m}{m^{\alpha'/2}} \right) - \exp\{-(\text{const.})(\ln m)^2\} \right\} = \\ = E_{m_1} \prod_{m=m_1}^{m_0} P\left( \max_{0 \leq h \leq 2m} Y_{t_{m,h}} \leq u_n - \frac{\ln m}{m^{\alpha'/2}} \right) \end{aligned} \quad (24)$$

where

$$E_{m_1} = \prod_{m=m_1}^{m_0} \left\{ \frac{1 - \exp\{-(\text{const.})(\ln m)^2\}}{P\left( \max_{0 \leq h \leq 2m} Y_{t_{m,h}} \leq u_n - \frac{\ln m}{m^{\alpha'/2}} \right)} \right\}.$$

We have established before that  $\max_{0 \leq h \leq 2m} Y_{t_{m,h}}$  is finite with probability 1 for all  $m$ . Hence

$$P\left( \max_{0 \leq h \leq 2m} Y_{t_{m,h}} \leq u_n - \frac{\ln m}{m^{\alpha'/2}} \right) \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

and

$$E_{m_1} \geq \exp\left\{-(\text{const.}) \sum_{m=m_1}^{\infty} e^{-(\ln m)^{3/2}}\right\}.$$

The R.H.S. above tends to one as  $n \rightarrow \infty$ . It remains to show that the product in R.H.S. in (24) is bounded below by the required limit. Now, since  $t_{m,h}$  and  $t_{m,j}$  are sufficiently far apart for  $h \neq j$ , we can write

$$\mathbf{E} Y_k Y_l \geq 1 - \left(\frac{1}{2} + \varepsilon'\right) \frac{L(l-k)}{L(k)} \left(\frac{l-k}{k}\right)^{\alpha}$$

for values of  $k$  and  $l$  in the set  $\tau_m = \{t_{m,h}, h = 0, 1, \dots, 2m\}$ . Notice that for such values of  $k$  and  $l$ ,  $(l-k)/k$  cannot tend to zero very rapidly i.e.,  $(l-k)/k^{1-\gamma} \rightarrow \infty \forall \gamma > 0$ . Hence applying (11(ii)) we have  $L(l-k)/L(l) \rightarrow 1$ . Also, since  $l \leq (1+\varepsilon)k$ ,  $L(l)/L(k) \rightarrow 1$  as well. By choosing an appropriate value of  $\varepsilon'$ , we can write for large  $k$  that

$$\mathbf{E} Y_k Y_l \geq 1 - \left(\frac{1}{2} + \varepsilon'\right) \left(\frac{l-k}{k}\right)^{\alpha}$$

for  $k, l \in \tau_m$ . Let  $\{\xi(t), t \geq 0\}$  be standard stationary Gaussian process with covariance function  $1 - (\frac{1}{2} + \varepsilon')|t|^\alpha$  for  $0 \leq t \leq \varepsilon$ . We see that the covariance matrix of  $\{Y_l, l \in \tau_m\}$  is bounded below by that of  $\{\xi_{(l-t_m)/t_m}, l \in \tau_m\}$ . Thus  $\max_{0 \leq h \leq 2m} Y_{t_m, h}$  is stochastically bounded above by  $\max_{0 \leq h \leq 2m} \xi_{(t_m, h - t_m)/t_m}$  which is at most  $\max_{0 \leq t \leq \varepsilon} \xi(t)$ . Thus the product in the R.H.S. of (24) is asymptotically at least

$$\begin{aligned} \exp\left\{-\sum_{m=m_1}^{m_0} P\left(\max_{0 \leq t \leq \varepsilon} \xi(t) > u_n - \frac{\ln m}{m^{\alpha'/2}}\right)\right\} &\sim \\ &\sim \exp\left\{-\sum_{m=m_1}^{m_0} \varepsilon \left(u_n - \frac{\ln m}{m^{\alpha'/2}}\right)^{2/\alpha-1} \phi\left(u_n - \frac{\ln m}{m^{\alpha'/2}}\right) \left(\frac{1}{2} + \varepsilon'\right)^{1/\alpha} H_\alpha\right\} \end{aligned} \quad (25)$$

by Lemma (3.8) of [9].  $H_\alpha$  is defined in the statement of Theorem 1. Substituting the value for  $u_n$ , we have the R.H.S. of (25) asymptotically equal to

$$\exp\left\{-\frac{\varepsilon}{\ln m} (1 + 2\varepsilon')^{1/\alpha} \exp(-x + o(1)) \sum_{m=m_1}^{m_0} \exp(u_n \ln m / m^{\alpha'/2})\right\}. \quad (26)$$

Splitting the sum into two parts say  $m_1 \leq m \leq (m_0)^{1/2}$  and  $m_0^{1/2} < m \leq m_0$ , we see that for large  $n$

$$\begin{aligned} \sum_{m=m_1}^{m_0} \exp(u_n \ln m / m^{\alpha'/2}) &\leq m_0^{1/2} \exp(u_n) + m_0 \exp(1/(\ln m)^{\alpha'/2}) \\ &= m_0(1 + o(1)). \end{aligned}$$

Substituting in (26) we get that

$$\liminf_{n \rightarrow \infty} P\left(\mu_n \leq \beta_n + \frac{x}{\chi_n}\right) \geq \exp(-(1 + 2\varepsilon')^{1/\alpha} e^{-x})$$

for all  $\varepsilon' > 0$ . This concludes the proof of (18).

It is much easier to show the other side, viz,

$$\limsup_{n \rightarrow \infty} P\left(\mu_n \leq \beta_n + \frac{x}{\chi_n}\right) \leq \exp(-e^{-x}). \quad (27)$$

We just exclude the appropriate number of  $Y_k$  and then compare them with variables selected from an appropriate  $\alpha$ -process by Berman's Lemma [1]. Let  $u_n$ ,  $m_0$  and  $m_1$  be the same as defined before. We will define new subsequences  $t_{m,h}$  and the sets  $\tau_m$ . To avoid clumsy notation, we will use the same symbols. Thus, it should be noted that  $t_{m,h}$  and  $\tau_m$  are defined two different ways in this section and two different ways in Section 3. All efforts to try to define a subsequence that will work for both parts of the proof in each section have failed so far. Care should be taken in noting an appropriate definition of  $t_{m,h}$  and  $\tau_m$  in reading different parts of the proof. Let

$$t_{m,h} = \left\lceil \exp\left\{\varepsilon \left(m + \frac{h}{(\ln m)^{2/\alpha}}\right)\right\} \right\rceil$$

and

$$\tau_m = \{t_{m,h}; 0 \leq h \leq [(1-\varepsilon)(\ln m)^{2/\alpha}] = m_h, \text{ say}\}.$$

(i.e., we have clipped a small portion from the right-hand side for each of the sets  $\tau_m$ ).  
Now

$$\begin{aligned} P(\mu_n \leq u_n) &\leq P\left\{\bigcap_{m=m_1}^{m_0} \left(\max_{t_{m,h} \in \tau_m} Y_{t_{m,h}} \leq u_n\right)\right\} \\ &\leq \prod_{m=m_1}^{m_0} P\left(\max_{t_{m,h} \in \tau_m} Y_{t_{m,h}} \leq u_n\right) \\ &\quad + C \sum_{P>q=m_1}^{m_0} \sum_{l=1}^{m_0} \sum_{h=1}^{q_h} \rho_{Plqh} \exp\left(-\frac{u_n^2}{1+\rho_{Plqh}}\right). \end{aligned} \quad (28)$$

The last inequality follows from Berman's Lemma.  $C$  is some positive constant and

$$\rho_{Plqh} = \begin{cases} \mathbf{E}(Y_{t_{P,l}} Y_{t_{q,h}}), & \text{if } P \neq q, \\ 0, & \text{if } P = q. \end{cases}$$

Notice also that the largest value of  $(1-\rho_{Plqh}^2)^{-1/2}$  depends only on  $\varepsilon$  and is absorbed in  $C$ . By Lemma 1,

$$\begin{aligned} \mathbf{E} Y_{t_{P,l}} Y_{t_{q,h}} &\leq 1 - (\tfrac{1}{2} - \varepsilon') \frac{v(t_{P,l} - t_{q,h})}{v(t_{q,h})} \\ &\sim 1 - (\tfrac{1}{2} - \varepsilon') \frac{L(t_{P,l} - t_{q,h})}{L(t_{q,h})} \left(\frac{t_{P,l} - t_{q,h}}{t_{q,h}}\right)^\alpha. \end{aligned}$$

Because of the above definitions, when  $P \neq q$ ,

$$t_{P,l} - t_{q,h} \geq t_{q,h}(e^{\varepsilon^2} - 1) \sim \varepsilon^2 t_{q,h}.$$

By the same reasoning as before and condition (11), we have  $L(t_{P,l} - t_{q,h})/L(t_{q,h}) \sim 1$ .  
Thus

$$\mathbf{E} Y_{t_{P,l}} Y_{t_{q,h}} \leq 1 - \varepsilon' \quad (29)$$

for all  $P \neq q$  and  $q$  large. For all  $m$ ,  $m_h \leq (\ln m_0)^{2/\alpha}$ . Substituting, an upperbound for the sum in the R.H.S. of (28) is

$$\begin{aligned} &c \sum_{q=m_1}^{m_0} \left\{ m_0^\eta (\ln m_0)^{4/\alpha} \exp\left(-\frac{u_n^2}{2-\varepsilon'}\right) \right. \\ &\quad \left. + \sum_{P=q+m_0}^{m_0} (\ln m_0)^{4/\alpha} \rho_{Plqh} \exp\left(-\frac{u_n^2}{1+\rho_{Plqh}}\right) \right\} \end{aligned} \quad (30)$$

for all  $0 < \eta < 1$ .  $u_n^2 \sim 2 \ln_2 n$  and  $m_0 \sim (\ln n)/\varepsilon$ . We choose  $\eta < \varepsilon'/(2-\varepsilon')$  for the value of  $\varepsilon'$  that works in (29). This makes the sum of the first term in (30) to be  $o(1)$ .

For the second part of the sum, we know by Lemma 2 that

$$\mathbf{E} Y_{t_{P,l}} Y_{t_{q,h}} \leq \left( \frac{v(t_{q,h})}{v(t_{P,l})} \right)^{1/2} + \left( \frac{t_{q,h}}{t_{P,l}} \right)^\gamma$$

for some  $\gamma > 0$ . But  $v(t_{q,h})/v(t_{P,l}) \sim (t_{q,h}/t_{P,l})^\alpha$  for large  $q$ . Thus for some  $\gamma' > 0$

$$\rho_{Plqh} \leq (t_{q,h}/t_{P,l})^{\gamma'} \leq \exp(-\gamma' \varepsilon m_0^n)$$

for all  $p \geq q + m_0^n$ . Thus the sum in (30) is  $o(1)$ . To bound the product in the R.H.S. of (28), we define a standard stationary Gaussian process  $\{\xi(t), 0 \leq t \leq \varepsilon\}$  (different from the one in the first part of the proof) with covariance function  $1 - (\frac{1}{2} - \varepsilon')|t|^\alpha$  near origin. We bound

$$P\left(\max_{t_{m,h} \in \tau_m} Y_{t_{m,h}} \leq u_n\right) \leq P\left(\max_{1 \leq h \leq m_h} \xi_{(t_{m,h}-t_m)/t_m} \leq u_n\right).$$

By Lemma (3.8) of [9], the R.H.S. above is asymptotically equal to

$$\varepsilon(1-\varepsilon)(1-\varepsilon')^{1/\alpha} H_\alpha u_n^{2/\alpha-1} \phi(u_n).$$

Substituting in the product we get the desired result (27). The details of the last argument are very much similar to those in the first part of the proof and hence are not repeated. This finishes the proof of Theorem 1.

### 3. Strongly dependent case

In Section 2 we considered the cases when  $v(n) \sim n^\alpha L(n)$  for  $0 < \alpha < 2$ . In this section we will have  $\alpha = 2$ . The sequence  $\{X_i, i \geq 0\}$  and  $\mu_n$  are as described in Section 2.

**Theorem 3.** Let  $f$  be a probability density function on the real line and set

$$A_k = \{(x, y) | -\infty < x < \infty; 0 \leq y < \infty \text{ and } f(x+k) > y\}.$$

Assume that the correlation function  $r_k = \mathbf{E} X_0 X_k$  satisfies

(i)

$$r_k = \int_{-\infty}^{\infty} f(x) \wedge f(x+k) dx,$$

(ii)

$$\text{for } k \geq k_0, \quad A_k \cap A_0 \subseteq A_l \cap A_0 \quad \text{for } 0 \leq k \leq l.$$

(31)

Let  $r_k \rightarrow 0$  be a slowly varying function such that for large  $k$

$$\sum_{i=-k}^k r_i = (2 + E_k) k r_k \quad (32)$$

with  $E_k = o(1/\ln k)^\theta$  nonincreasing and  $\theta > 0$ . Then

$$\lim_{n \rightarrow \infty} P\left(\mu_n \leq \beta_n + \frac{x}{\chi_n}\right) = \exp(-e^{-x}) \quad (33)$$

for all  $-\infty < x < \infty$ . Here

$$\chi_n = (2 \ln_2 1/r_n)^{1/2} \quad \text{and} \quad \beta_n = \chi_n + \frac{1}{2} \frac{\ln_2 1/r_n - \ln 4\pi}{\chi_n}.$$

Condition (31) is used and discussed in Mittal and Ylvisaker [7]. From discussion there, we have

$$X_i = (1 - r_k)^{1/2} W_i^k + I_k, \quad 0 \leq i \leq k$$

where  $W_i^k$  and  $I_k$  are independent normal with mean zero,  $\text{Var}(W_i^k) = 1$  and  $\mathbb{E}I_k I_l = r_k, \forall l \geq k$ . Thus

$$Y_k = \frac{\sum_{i=1}^k X_i}{(v(k))^{1/2}} = (1 - r_k) \frac{\sum_{i=1}^k W_i^k}{(v(k))^{1/2}} + \frac{k I_k}{(v(k))^{1/2}}.$$

The general idea of the proof is to show that the variables  $\sum_{i=1}^k W_i^k / (v(k))^{1/2}$  are too small and hence we can replace the process  $\{Y_k\}$  by  $\{k I_k / (v(k))^{1/2}\}$ . (In fact, we replace it by  $\eta_k = I_k / (r_k)^{1/2}$ . We deal with the new process in very much similar ways as in Section 2. The following lemma does the first part of the proof.

**Lemma 3.** Under the conditions of Theorem 3,

$$\lim_{k \rightarrow \infty} (\ln k)^\delta \zeta_k = 0 \text{ a.s.} \quad (34)$$

for all  $\delta > 0$  sufficiently small, where  $\zeta_k = Y_k - \eta_k$ .

**Proof.** Let us define  $t_k = [\exp(k^\gamma)]$  for  $0 < \gamma < 1$  and

$$A_k = A_k(\varepsilon) = \left\{ \max_{t_k \leq j < t_{k+1}} \zeta_j > \varepsilon \right\}$$

for  $\varepsilon > 0, k = 0, 1, \dots$ . The result follows by symmetry and the use of Borel–Cantelli Lemma if we show that  $\sum_{k=1}^{\infty} P(A_k) < \infty$ . To compute  $P(A_k)$ , we find lower bound on the correlations of  $\zeta_j, t_k \leq j \leq t_{k+1}$ . We first note that as a result of the assumed conditions,  $r_k \geq 0$  and for sufficiently large  $k, r_k \leq r_i \forall i \leq k$ . Now for  $j \leq l$ ,

$$\mathbb{E} \zeta_l \zeta_j = \mathbb{E} Y_l Y_j - \frac{j r_j + \sum_{i=j+1}^l r_i}{(r_j v(l))^{1/2}} - \frac{j (r_l)^{1/2}}{(v(j))^{1/2}} + \left(\frac{r_l}{r_j}\right)^{1/2}. \quad (35)$$

But

$$r_l^{1/2} \left\{ \frac{1}{r_j^{1/2}} - \frac{j}{(v(j))^{1/2}} \right\} = r_l^{1/2} \left\{ \frac{(v(j))^{1/2} - j r_j^{1/2}}{(r_j v(j))^{1/2}} \right\} \sim \left(\frac{r_l}{r_j}\right)^{1/2} E_j \geq 0 \quad \text{for large } j$$

and

$$\begin{aligned} \mathbf{E} Y_l Y_j - \frac{j r_j + \sum_{i+1}^l r_i}{(r_l v(l))^{1/2}} &> \left( \frac{v(j)}{v(l)} \right)^{1/2} - \frac{l r_j^{1/2}}{(v(l))^{1/2}} = \frac{(v(j))^{1/2} - l r_j}{(v(l))^{1/2}} \\ &\sim \frac{j r_j^{1/2}}{l r_l^{1/2}} \left( \mathbf{E}_j - \frac{l-j}{j} \right). \end{aligned}$$

For  $t_k \leq l, j < t_{k+1}$ ,  $(l-j)/j \leq (\text{const.}) k^{-(1-\gamma)}$ . Thus for large  $j$ , since  $r_j \geq r_l$ ,

$$\mathbf{E} \zeta_l \zeta_j \geq E_j \frac{j}{l} \geq E_j \left( 1 - \frac{c_1}{k^{1-\gamma}} \right)$$

where  $c > 0$  is a constant. We know that  $v(Z_j) = 2(1 - j r_j^{1/2}/(v(j))^{1/2}) \sim E_j$  and  $E_j$  is nonincreasing. Hence

$$\mathbf{E} \frac{\zeta_l \zeta_j}{(v(Z_j) v(Z_l))^{1/2}} \geq 1 - c_1 k^{-(1-\gamma)} \quad \text{for } t_k \leq j, l < t_{k+1}$$

Now,

$$\begin{aligned} P(A_k) &= P\left( \max_{t_k \leq j < t_{k+1}} \zeta_j > \varepsilon \right) \\ &\leq P\left( \max_{t_k \leq j < t_{k+1}} \frac{\zeta_j}{(v(Z_j))^{1/2}} > \frac{\varepsilon}{(E_{t_k})^{1/2}} \right) \\ &\leq P\left\{ \left( \frac{c_1}{k^{1-\gamma}} \right)^{1/2} M_{t_{k+1}-t_k}^* + \left( 1 - \frac{c_1}{k^{1-\gamma}} \right)^{1/2} U > \varepsilon k^{\gamma\theta/2} \right\} \end{aligned} \quad (36)$$

where  $M_n^*$  is maximum of  $n$  independent standard normal variables and  $U$  also standard normal, independent of  $M_n^*$ . An upper bound for the R.H.S. of (36) is

$$P\left( M_{t_{k+1}-t_k}^* > \frac{\varepsilon}{2c_1^{1/2}} k^{(1-\gamma)/2 + \gamma\theta/2} \right) + 1 - \Phi\left( \frac{\varepsilon}{2} k^{\gamma\theta/2} \right)$$

where  $\Phi(x) = \int_{-\infty}^x \phi(u) du$ . Noticing that  $t_{k+1} - t_k \sim \exp(k^\gamma)$ , we see that the above is summable over  $k$ . This completes the proof of the lemma.

Next, we can replace  $\mu_n$  in (33) by  $\nu_n = \max_{0 \leq k \leq n} I_k / r_k^{1/2} =: \max_{0 \leq k \leq n} \eta_k$  because

$$\chi_n |\mu_n - \nu_n| \leq \chi_n \max_{0 \leq k \leq n} \zeta_k$$

and

$$\frac{\chi_n}{(\ln n)^\delta} \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{for all } \delta > 0.$$

For the proof of (33) with  $\nu_n$  in place of  $\mu_n$ , we follow the same procedure as in Section 2. We compare  $\{\eta_k\}$  with a sequence sampled from an appropriately chosen

( $\alpha = 1$ ) process. Unlike Section 2, the frequency of sampling here depends on the covariance function  $\{r_k\}$ . Some of the details of Section 2 become easier here in view of the observation that  $I_k = B(r_k)$  where  $B$  denotes a standardized Brownian motion. Sketch of the proof is given below avoiding repetition of arguments in Section 2.

Define (different from all previous definitions)

$$t_k = [r^{-1}(e^{-\varepsilon k})]; \quad t_{m,k} = [r^{-1}(e^{-\varepsilon(m^2+k)^{1/2}})]$$

and

$$\tau_m = \{t_{m,k}; 0 \leq k \leq 2m\}.$$

Here  $r^{-1}(\delta) = \min_t(r(t) = \delta)$ . Since  $r$  is monotone,  $t_k$  is nondecreasing. Also  $r(r^{-1}(\delta)) = \delta$ . By similar arguments that lead to (29) we have

$$P(\nu'_n \leq u_n) \geq \prod_{m=m_1}^{m_0} P\left(\max_{t_{m,k} \in \tau_m} \eta_{t_{m,k}} \leq u_n\right) \quad (37)$$

where  $u_n = \beta_n + x/\chi_n$ ;  $m_0$  and  $m_1$  are such that  $t_{m_1} \sim (\ln n)^{1/2}$  and  $m_0 = \min\{m \mid t_m \geq n\}$ . Also  $\nu'_n = \max_{(\ln n)^{1/2} \leq k \leq n} \eta_k$ . Define  $Z_l = \eta_l - \eta_{t_{m,k-1}}$  for  $t_{m,k} \leq l < t_{m,k+1}$ . Then

$$P\left(\max_{t_m \leq l < t_{m+1}} \eta_l \leq u_n\right) \geq P\left(\max_{0 \leq k \leq 2m} \eta_{t_{m,k}} \leq u_n - \frac{\ln m}{m^{1/2}}\right) \\ - P\left(\max_{t_m \leq l < t_{m+1}} Z_l > \frac{\ln m}{m^{1/2}}\right).$$

But

$$P\left(\max_{t_m \leq l < t_{m+1}} Z_l > \frac{\ln m}{m^{1/2}}\right) \leq (2m+1)P\left(\max_{t_{m,k} \leq l < t_{m,k+1}} \frac{I_l}{r_l^{1/2}} - \frac{I_{t_{m,k+1}}}{r_{t_{m,k+1}}^{1/2}} > \frac{\ln m}{m^{1/2}}\right) \\ \leq (2m+1)\left\{P\left(\max_{t_{m,k} \leq l < t_{m,k+1}} \frac{I_l - I_{t_{m,k+1}}}{r_l^{1/2}} > \frac{\ln m}{m^{1/2}}\right) \right. \\ \left. + P\left(\max_{t_{m,k} \leq l < t_{m,k+1}} I_{t_{m,k+1}} \left(r_l^{-1/2} - r_{t_{m,k+1}}^{-1/2}\right) > \frac{\ln m}{m^{1/2}}\right)\right\} \\ \leq (2m+1)\left\{P\left(\max_{t_{m,k} \leq l < t_{m,k+1}} (I_l - I_{t_{m,k+1}}) > \frac{\ln m}{m^{1/2} r_{t_{m,k+1}}^{1/2}}\right) \right. \\ \left. + P\left(I_{t_{m,k+1}} (r_{t_{m,k+1}}^{-1/2} - r_{t_{m,k}}^{-1/2}) > \frac{\ln m}{m^{1/2}}\right)\right\}. \quad (38)$$

Since  $I_l = B(r_l)$ , we can write  $I_l - I_{t_{m,k+1}}$  as  $B(r_l - r_{t_{m,k+1}})$ . Using Proposition 12.20 of Brieman [2], we get the first probability in the R.H.S. of (38) to be at most

$$2P\left(I_{t_{m,k}} - I_{t_{m,k+1}} > \frac{\ln m}{m^{1/2} r_{t_{m,k+1}}^{1/2}}\right). \quad (39)$$



But  $r_{t_{m,k}} \sim \exp(-\varepsilon(m^2 + k)^{1/2})$ . Thus

$$\begin{aligned} \text{Var}(I_{t_{m,k}} - I_{t_{m,k+1}}) &= r_{t_{m,k}} - r_{t_{m,k+1}} \\ &\sim \exp(-\varepsilon(m^2 + k + 1)^{1/2})[e^{-\varepsilon/2m} - 1] \sim \frac{\varepsilon}{2m} r_{t_{m,k+1}} \end{aligned}$$

and (39) is at most  $\varepsilon\phi(2(\ln m)/\varepsilon)/\ln m$ . Substituting values for  $r_{t_{m,k}}$ , we get approximately the same bound for the second probability in the R.H.S. of (38). Hence

$$P\left(\max_{t_m \leq l < t_{m+1}} Z_l > \frac{\ln m}{m^{1/2}}\right) \leq \frac{2(2m+1)\phi\left(\frac{2 \ln m}{\varepsilon}\right)}{\ln m}. \quad (40)$$

The R.H.S. of (40) is summable over  $m$ . Following the same line as in Section 2, it only remains to show that

$$P\left(\max_{0 \leq k \leq 2m} \eta_{t_{m,k}} \leq u_n\right) \sim \varepsilon u_n \phi(u_n) \left(\frac{1}{2} + \varepsilon'\right).$$

(Notice that the constant  $H_\alpha$  for  $\alpha = 1$  is explicitly given in Theorem 4.4 of [8]). Let  $\{\xi(t), t \geq 0\}$  be a standard stationary Gaussian process with covariance function  $1 - (\frac{1}{2} + \varepsilon')|t|$  near origin. Then

$$\begin{aligned} \mathbf{E}(\eta_{t_{m,k+l}} \eta_{t_{m,k}}) &= (r_{t_{m,k+l}}/r_{t_{m,k}})^{1/2} \\ &\sim \exp\left\{\frac{\varepsilon}{2}((m^2 + k + l)^{1/2} - (m^2 + k)^{1/2})\right\} \\ &\geq 1 - \frac{(1 + \varepsilon)\varepsilon l}{4(m^2 + k)^{1/2}} \geq 1 - \left(\frac{1}{2} + \varepsilon\right)\left(\frac{\varepsilon l}{2m}\right) = \mathbf{E}(\xi_{(t_{m,l}-t_m)/t_m} \xi_0). \end{aligned}$$

It follows in the manner similar to Section 2 that

$$\lim_{n \rightarrow \infty} P\left(\mu_n \leq \beta_n + \frac{x}{\chi_n}\right) \geq \exp(-e^{-x}). \quad (41)$$

(It should be noted that  $r_{t_{m0}} \sim \exp(-\varepsilon m_0) \geq e^{-\varepsilon} r_n$ . Strictly speaking, the normalized constants for the  $\xi$  process should be with  $r_n$  replaced by  $r_{t_{m0}}$  but it is easy to see that this does not make any difference in the limit.)

The procedure for the rest of the proof is now apparent. We define a new

$$t_{m,k} = \left\lceil r^{-1} \left\{ \exp\left(-\varepsilon\left(m + \frac{k}{(\ln m^2)}\right)\right) \right\} \right\rceil.$$

The fact that  $\rho_{Plqh} = \mathbf{E}\eta_{t_{p,l}}\eta_{t_{q,h}}$  decreases rapidly to zero for  $p - q \geq m_0$  is quite obvious. The rest of the arguments are very similar and hence not repeated.

#### 4. Examples and discussion

Examples of covariance functions  $\{r_k\}$  satisfying (10) and (11) are given by convex  $\{r_k\}$ ,  $r_k = k^{-\gamma}(\ln k)^l$  for  $k \geq k_0$ ,  $0 < \gamma < 1$  and  $l \in \mathbf{R}$ . (Defined appropriately on  $[0, k_0]$  to make it convex). For  $\{r_k\}$  satisfying (31) and (32), we give again convex  $\{r_k\}$ ,  $r_k = (\ln k)^{-1}$  for  $k \geq k_0$  and  $l > 0$ .

The proof of the result  $\lim_{n \rightarrow \infty} P(\nu_n \leq \beta_n + x/\chi_n) = \exp(-e^{-x})$  can be greatly reduced by the observation  $I_k/r_k^{1/2} = B(r_k)/r_k^{1/2}$  and the result of Darling and Erdős [3] as given in (3.1) of Robbins and Siegmund [10]. Darling and Erdős do not state their result in the form (3.1) of [10]. Also their proof of Theorem 1 in [3] is quite complicated. Theorem 1 of [3] is a particular case of Theorem 1 and the last part of Theorem 3 offers an alternative proof for (3.1) of [10].

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